

HARDY-LITTLEWOOD-PALEY INEQUALITIES AND FOURIER MULTIPLIERS ON $SU(2)$

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ABSTRACT. In this paper we prove noncommutative versions of Hardy–Littlewood and Paley inequalities relating a function and its Fourier coefficients on the group $SU(2)$. As a consequence, we use it to obtain lower bounds for the L^p – L^q norms of Fourier multipliers on the group $SU(2)$, for $1 < p \leq 2 \leq q < \infty$. In addition, we give upper bounds of a similar form, analogous to the known results on the torus, but now in the noncommutative setting of $SU(2)$.

1. INTRODUCTION

Let \mathbb{T}^n be the n -dimensional torus and let $1 < p \leq q < \infty$. A sequence $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}^n}$ of complex numbers is said to be a multiplier of trigonometric Fourier series from $L^p(\mathbb{T}^n)$ to $L^q(\mathbb{T}^n)$ if the operator

$$T_\lambda f(x) = \sum_{k \in \mathbb{Z}^n} \lambda_k \widehat{f}(k) e^{ikx}$$

is bounded from $L^p(\mathbb{T}^n)$ to $L^q(\mathbb{T}^n)$. We denote by \mathbf{m}_p^q the set of such multipliers.

Many problems in harmonic analysis and partial differential equations can be reduced to the boundedness of multiplier transformations. There arises a natural question of finding sufficient conditions for $\lambda \in \mathbf{m}_p^q$. The topic of \mathbf{m}_p^q multipliers has been extensively researched. Using methods such as the Littlewood–Paley decomposition and Calderon–Zygmund theory, it is possible to prove Hörmander–Mihlin type theorems, see e.g. Mihlin [Mih57, Mih56], Hörmander [Hör60], and later works.

Multipliers have been then analysed in a variety of different settings, see e.g. Gaudry [Gau66], Cowling [Cow74], Vretare [Vre74]. The literature on the spectral multipliers is too rich to be reviewed here, see e.g. a recent paper [CKS11] and references therein. The same is true for multipliers on locally compact abelian groups, see e.g. [Arh12], or for Fourier or spectral multipliers on symmetric spaces, see e.g. [Ank90] or [CGM93], resp. We refer to the above and to other papers for further references on the history of \mathbf{m}_p^q multipliers on spaces of different types.

In this paper we are interested in questions for Fourier multipliers on compact Lie groups, in which case the literature is much more sparse: in the sequel we will make a more detailed review of the existing results. Thus, in this paper we will be investigating several questions in the model case of Fourier multipliers on the compact group $SU(2)$. Although we will not explore it in this paper, we note that there are

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links between multipliers on $SU(2)$ and those on the Heisenberg group, see Ricci and Rubin [RR86].

In general, most of the multiplier theorems imply that $\lambda \in \mathbf{m}_p^p$ for all $1 < p < \infty$ at once. In [Ste70], Stein raised the question of finding more subtle sufficient conditions for a multiplier to belong to some \mathbf{m}_p^p , $p \neq 2$, without implying also that it belongs to all \mathbf{m}_p^p , $1 < p < \infty$. In [NT00], Nursultanov and Tleukhanova provided conditions on $\lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$ to belong to \mathbf{m}_p^q for the range $1 < p \leq 2 \leq q < \infty$. In particular, they established lower and upper bounds for the norms of multiplier $\lambda \in \mathbf{m}_p^q$ which depend on parameters p and q . Thus, this provided a partial answer to Stein's question. Let us recall this result in the case $n = 1$:

Theorem 1.1. *Let $1 < p \leq 2 \leq q < \infty$ and let M_0 denote the set of all finite arithmetic sequences in \mathbb{Z} . Then the following inequalities hold:*

$$\sup_{Q \in M_0} \frac{1}{|Q|^{1+\frac{1}{q}-\frac{1}{p}}} \left| \sum_{m \in Q} \lambda_m \right| \lesssim \|T_\lambda\|_{L^p \rightarrow L^q} \lesssim \sup_{k \in \mathbb{N}} \frac{1}{k^{1+\frac{1}{q}-\frac{1}{p}}} \sum_{m=1}^k \lambda_m^*,$$

where λ_m^* is a non-increasing rearrangement of λ_m , and $|Q|$ is the number of elements in the arithmetic progression Q .

In this paper we study the noncommutative versions of this and other related results. As a model case, we concentrate on analysing Fourier multipliers between Lebesgue spaces on the group $SU(2)$ of 2×2 unitary matrices with determinant one. Sufficient conditions for Fourier multipliers on $SU(2)$ to be bounded on L^p -spaces have been analysed by Coifman-Weiss [CW71b] and Coifman-de Guzman [CdG71], see also Chapter 5 in Coifman and Weiss' book [CW71a], and are given in terms of the Clebsch-Gordan coefficients of representations on the group $SU(2)$. A more general perspective was provided in [RW13] where conditions on Fourier multipliers to be bounded on L^p were obtained for general compact Lie groups.

Results about spectral multipliers are more known, for functions of the Laplacian (N. Weiss [Wei72] or Coifman and Weiss [CW74]), or of the sub-Laplacian on $SU(2)$, see Cowling and Sikora [CS01]. However, following [CW71b, CW71a, RW13], here we are rather interested in Fourier multipliers.

In this paper we obtain lower and upper estimates for the norms of Fourier multipliers acting between L^p and L^q spaces on $SU(2)$. These estimates explicitly depend on parameters p and q . Thus, this paper can be regarded as a contribution to Stein's question in the noncommutative setting of $SU(2)$. At the same time we provide a noncommutative analogue of Theorem 1.1. Briefly, let A be the Fourier multiplier on $SU(2)$ given by

$$\widehat{A}f(l) = \sigma_A(l)\widehat{f}(l), \text{ for } \sigma_A(l) \in \mathbb{C}^{(2l+1) \times (2l+1)}, \quad l \in \frac{1}{2}\mathbb{N}_0,$$

where we refer to Section 1 for definitions and notation related to the Fourier analysis on $SU(2)$. For such operators, in Theorem 3.1, for $1 < p \leq 2 \leq q < \infty$, we give two lower bounds, one of which is of the form

$$(1.1) \quad \sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{1}{(2l+1)^{1+\frac{1}{q}-\frac{1}{p}}} \left(\frac{1}{2l+1} |\text{Tr } \sigma_A(l)| \right) \lesssim \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))}.$$

A related upper bound

$$(1.2) \quad \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))} \lesssim \sup_{s>0} s \left(\sum_{\substack{l \in \frac{1}{2}\mathbb{N}_0 \\ \|\sigma_A(l)\|_{op} \geq s}} (2l+1)^2 \right)^{\frac{1}{p} - \frac{1}{q}}.$$

will be given in Theorem 4.1.

The proof of the lower bound is based on the new inequalities describing the relationship between the “size” of a function and the “size” of its Fourier transform. These inequalities can be viewed as a noncommutative $SU(2)$ -version of the Hardy-Littlewood inequalities obtained by Hardy and Littlewood in [HL27]. To explain this briefly, we recall that in [HL27], Hardy and Littlewood have shown that for $1 < p \leq 2$ and $f \in L^p(\mathbb{T})$, the following inequality holds true:

$$(1.3) \quad \sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}(m)|^p \leq K \|f\|_{L^p(\mathbb{T})}^p,$$

arguing this to be a suitable extension of the Plancherel identity to L^p -spaces. While we refer to Section 1 and to Theorem 2.1 for more details on this, our analogue for this is the inequality

$$(1.4) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)(2l+1)^{\frac{5}{2}(p-2)} \|\widehat{f}(l)\|_{\text{HS}}^p \leq c \|f\|_{L^p(SU(2))}^p, \quad 1 < p \leq 2,$$

which for $p = 2$ gives the ordinary Plancherel identity on $SU(2)$, see (2.1). We refer to Theorem 2.2 for this and to Corollary 2.3 for the dual statement. For $p \geq 2$, the necessary conditions for a function to belong to L^p are usually harder to obtain. In Theorem 2.8 we give such a result for $2 \leq p < \infty$ which takes the form

$$(1.5) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{p-2} \left(\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \geq l}} \frac{1}{2k+1} \left| \text{Tr } \widehat{f}(k) \right| \right)^p \leq c \|f\|_{L^p(SU(2))}^p, \quad 2 \leq p < \infty.$$

In turn, this gives a noncommutative analogue to the known similar result on the circle (which we recall in Theorem 2.7). Similar to (1.1), the averaged trace appears also in (1.5) – it is the usual trace divided by the number of diagonal elements in the matrix.

In [Hör60] Hörmander proved a Paley-type inequality for the Fourier transform on \mathbb{R}^N . In this paper we obtain an analogue of this inequality on the group $SU(2)$.

The results on the group $SU(2)$ are usually quite important since, in view of the resolved Poincaré conjecture, they provide information about corresponding transformations on general closed simply-connected three-dimensional manifolds (see [RT10] for a more detailed outline of such relations). In our context, they give explicit versions of known results on the circle \mathbb{T} or on the torus \mathbb{T}^n , in the simplest noncommutative setting of $SU(2)$.

At the same time, we note that some results of this paper can be extended to Fourier multipliers on general compact Lie groups. However, such analysis requires a more abstract approach, and will appear elsewhere.

The paper is organised as follows. In Section 2 we fix the notation for the representation theory of $SU(2)$ and formulate estimates relating functions with its Fourier coefficients: the $SU(2)$ -version of the Hardy–Littlewood and Paley inequalities and further extensions. In Section 3 we formulate and prove the lower bounds for operator norms of Fourier multipliers, and in Section 4 the upper bounds. Our proofs are based on inequalities from Section 2. In Section 5 we complete the proofs of the results presented in previous sections.

We shall use the symbol C to denote various positive constants, and $C_{p,q}$ for constants which may depend only on indices p and q . We shall write $x \lesssim y$ for the relation $|x| \leq C|y|$, and write $x \cong y$ if $x \lesssim y$ and $y \lesssim x$.

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2. HARDY-LITTLEWOOD AND PALEY INEQUALITIES ON $SU(2)$

The aim of this section is to discuss necessary conditions and sufficient conditions for the $L^p(SU(2))$ -integrability of a function by means of its Fourier coefficients. The main results of this section are Theorems 2.2, 2.4 and 2.8. These results will provide a noncommutative version of known results of this type on the circle \mathbb{T} . The proofs of most of the results of this Section are given in Section 5.

First, let us fix the notation concerning the representations of the compact Lie group $SU(2)$. There are different types of notation in the literature for the appearing objects - we will follow the notation of Vilenkin [Vil68], as well as that in [RT10, RT13]. Let us identify $z = (z_1, z_2) \in \mathbb{C}^{1 \times 2}$, and let $\mathbb{C}[z_1, z_2]$ be the space of two-variable polynomials $f: \mathbb{C}^2 \rightarrow \mathbb{C}$. Consider mappings

$$t^l: SU(2) \rightarrow GL(V_l), \quad (t^l(u)f)(z) = f(zu),$$

where $l \in \frac{1}{2}\mathbb{N}_0$ is called the quantum number, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and where V_l is the $(2l+1)$ -dimensional subspace of $\mathbb{C}[z_1, z_2]$ containing the homogeneous polynomials of order $2l \in \mathbb{N}_0$, i.e.

$$V_l = \{f \in \mathbb{C}[z_1, z_2]: f(z_1, z_2) = \sum_{k=0}^{2l} a_k z_1^k z_2^{2l-k}, \quad \{a_k\}_{k=0}^{2l} \subset \mathbb{C}\}.$$

The unitary dual of $SU(2)$ is

$$\widehat{SU(2)} \cong \{t^l \in \text{Hom}(SU(2), U(2l+1)): l \in \frac{1}{2}\mathbb{N}_0\},$$

where $U(d) \subset \mathbb{C}^{d \times d}$ is the unitary matrix group, and matrix components $t_{mn}^l \in C^\infty(SU(2))$ can be written as products of exponentials and Legendre-Jacobi functions, see Vilenkin [Vil68]. It is also customary to let the indices m, n to range from $-l$ to l , equi-spaced with step one. We define the Fourier transform on $SU(2)$ by

$$\widehat{f}(l) := \int_{SU(2)} f(u) t^l(u)^* du,$$

with the inverse Fourier transform (Fourier series) given by

$$f(u) = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \text{Tr} \left(\widehat{f}(l) t^l(u) \right).$$

The Peter-Weyl theorem on $SU(2)$ implies, in particular, that this pair of transforms are inverse to each other and that the Plancherel identity

$$(2.1) \quad \|f\|_{L^2(SU(2))}^2 = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \|\widehat{f}(l)\|_{\text{HS}}^2 =: \|\widehat{f}\|_{\ell^2(SU(2))}^2$$

holds true for all $f \in L^2(SU(2))$. Here $\|\widehat{f}(l)\|_{\text{HS}}^2 = \text{Tr} \left(\widehat{f}(l) \widehat{f}(l)^* \right)$ denotes the Hilbert-Schmidt norm of matrices. For more details on the Fourier transform on $SU(2)$ and on arbitrary compact Lie groups, and for subsequent Fourier and operator analysis we can refer to [RT10].

There are different ways to compare the “sizes” of f and \widehat{f} . Apart from the Plancherel’s identity (2.1), there are other important relations, such as the Hausdorff-Young or the Riesz-Fischer theorems. However, such estimates usually require the change of the exponent p in L^p -measurements of f and \widehat{f} . Our first results deal with comparing f and \widehat{f} in the same scale of L^p -measurements. Let us remark on the background of this problem. In [HL27, Theorems 10 and 11], Hardy and Littlewood proved the following generalisation of the Plancherel’s identity.

Theorem 2.1 (Hardy–Littlewood [HL27]). *The following holds.*

(1) *Let $1 < p \leq 2$. If $f \in L^p(\mathbb{T})$, then*

$$(2.2) \quad \sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}(m)|^p \leq K_p \|f\|_{L^p(\mathbb{T})}^p,$$

where K_p is a constant which depends only on p .

(2) *Let $2 \leq p < \infty$. If $\{\widehat{f}(m)\}_{m \in \mathbb{Z}}$ is a sequence of complex numbers such that*

$$(2.3) \quad \sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}(m)|^p < \infty,$$

then there is a function $f \in L^p(\mathbb{T})$ with Fourier coefficients given by $\widehat{f}(m)$, and

$$\|f\|_{L^p(\mathbb{T})}^p \leq K'_p \sum_{m \in \mathbb{Z}} (1 + |m|)^{p-2} |\widehat{f}(m)|^p.$$

Hewitt and Ross [HR74] generalised this theorem to the setting of compact abelian groups. Now, we give an analogue of the Hardy–Littlewood Theorem 2.1 in the noncommutative setting of the compact group $SU(2)$.

Theorem 2.2. *If $1 < p \leq 2$ and $f \in L^p(SU(2))$, then we have*

$$(2.4) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p-4} \|\widehat{f}(l)\|_{\text{HS}}^p \leq c_p \|f\|_{L^p(SU(2))}^p.$$

We can write this in the form more resembling the Plancherel identity, namely, as

$$(2.5) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)(2l+1)^{\frac{5}{2}(p-2)} \|\widehat{f}(l)\|_{\text{HS}}^p \leq c_p \|f\|_{L^p(SU(2))}^p,$$

providing a resemblance to both (2.2) and (2.1). By duality, we obtain

Corollary 2.3. *If $2 \leq p < \infty$ and $\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p-4} \|\widehat{f}(l)\|_{\text{HS}}^p < \infty$, then $f \in L^p(\text{SU}(2))$ and we have*

$$(2.6) \quad \|f\|_{L^p(\text{SU}(2))}^p \leq c_p \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p-4} \|\widehat{f}(l)\|_{\text{HS}}^p.$$

For $p = 2$, both of these statements reduce to the Plancherel identity (2.1).

In [Hör60] Hörmander proved a Paley-type inequality for the Fourier transform on \mathbb{R}^N . We now give an analogue of this inequality on the group $\text{SU}(2)$.

Theorem 2.4. *Let $1 < p \leq 2$. Suppose $\{\sigma(l)\}_{l \in \frac{1}{2}\mathbb{N}_0}$ is a sequence of complex matrices $\sigma(l) \in \mathbb{C}^{(2l+1) \times (2l+1)}$ such that*

$$(2.7) \quad K_\sigma := \sup_{s>0} s \sum_{\substack{l \in \frac{1}{2}\mathbb{N}_0 \\ \|\sigma(l)\|_{\text{op}} \geq s}} (2l+1)^2 < \infty.$$

Then we have

$$(2.8) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{p(\frac{2}{p}-\frac{1}{2})} \|\widehat{f}(l)\|_{\text{HS}}^p \|\sigma(l)\|_{\text{op}}^{2-p} \lesssim K_\sigma^{2-p} \|f\|_{L^p(\text{SU}(2))}^p.$$

It will be useful to recall the spaces $\ell^p(\widehat{\text{SU}(2)})$ on the discrete unitary dual $\widehat{\text{SU}(2)}$. For general compact Lie groups these spaces have been introduced and studied in [RT10, Section 10.3]. In the particular case of $\text{SU}(2)$, for a sequence of complex matrices $\sigma(l) \in \mathbb{C}^{(2l+1) \times (2l+1)}$ they can be defined by the finiteness of the norms

$$(2.9) \quad \|\sigma\|_{\ell^p(\widehat{\text{SU}(2)})} := \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{p(\frac{2}{p}-\frac{1}{2})} \|\sigma(l)\|_{\text{HS}}^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

and

$$(2.10) \quad \|\sigma\|_{\ell^\infty(\widehat{\text{SU}(2)})} := \sup_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{-\frac{1}{2}} \|\sigma(l)\|_{\text{HS}}.$$

Among other things, it was shown in [RT10, Section 10.3] that these spaces are interpolation spaces, they satisfy the duality property and, with $\sigma = \widehat{f}$, the Hausdorff-Young inequality

$$(2.11) \quad \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{p'(\frac{2}{p'}-\frac{1}{2})} \|\widehat{f}(l)\|_{\text{HS}}^{p'} \right)^{\frac{1}{p'}} \equiv \|\widehat{f}\|_{\ell^{p'}(\widehat{\text{SU}(2)})} \lesssim \|f\|_{L^p(\text{SU}(2))}, \quad 1 \leq p \leq 2.$$

Further, we recall a result on the interpolation of weighted spaces from [BL76]:

Theorem 2.5 (Interpolation of weighted spaces). *Let us write $d\mu_0(x) = \omega_0(x)d\mu(x)$, $d\mu_1(x) = \omega_1(x)d\mu(x)$, and write $L^p(\omega) = L^p(\omega d\mu)$ for the weight ω . Suppose that $0 < p_0, p_1 < \infty$. Then*

$$(L^{p_0}(\omega_0), L^{p_1}(\omega_1))_{\theta, p} = L^p(\omega),$$

where $0 < \theta < 1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\omega = w_0^{\frac{1-\theta}{p_0}} w_1^{\frac{\theta}{p_1}}$.

From this we obtain:

Corollary 2.6. *Let $1 < p \leq b \leq p' < \infty$. If $\{\sigma(l)\}_{l \in \frac{1}{2}\mathbb{N}_0}$ satisfies condition (2.7) with constant K_σ , then we have*

$$(2.12) \quad \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{b(\frac{2}{b}-\frac{1}{2})} \left(\|\widehat{f}(l)\|_{\text{HS}} \|\sigma(l)\|_{\text{op}}^{\frac{1}{b}-\frac{1}{p'}} \right)^b \right)^{\frac{1}{b}} \lesssim (K_\sigma)^{\frac{1}{b}-\frac{1}{p'}} \|f\|_{L^p(SU(2))}.$$

This reduces to (2.11) when $b = p'$ and to (2.8) when $b = p$.

Proof. We consider a sub-linear operator A which takes a function f to its Fourier transform $\widehat{f}(l)$ divided by $\sqrt{2l+1}$ i.e.

$$f \mapsto Af =: \left\{ \frac{\widehat{f}(l)}{\sqrt{2l+1}} \right\}_{l \in \frac{1}{2}\mathbb{N}_0},$$

where

$$\widehat{f}(l) = \int_{SU(2)} f(u) t^l(u)^* u \in \mathbb{C}^{(2l+1) \times (2l+1)}, \quad l \in \frac{1}{2}\mathbb{N}_0.$$

The statement follows from Theorem 2.5 if we regard the left-hand sides of inequalities (2.8) and (2.11) as an $\|Af\|_{L^p}$ -norm in a weighted sequence space over $\frac{1}{2}\mathbb{N}_0$ with the weights given by $w_0(l) = (2l+1)^2 \|\sigma(l)\|_{\text{op}}^{2-p}$ and $w_1(l) = (2l+1)^2$, $l \in \frac{1}{2}\mathbb{N}_0$. \square

Coming back to the Hardy–Littlewood Theorem 2.1, we see that the convergence of the series (2.3) is a sufficient condition for f to belong to $L^p(\mathbb{T})$, for $p \geq 2$. However, this condition is not necessary. Hence, there arises the question of finding necessary conditions for f to belong to L^p . In other words, there is the problem of finding lower estimates for $\|f\|_{L^p}$ in terms of the series of the form (2.3). Such result on $L^p(\mathbb{T})$ was obtained by Nursultanov and can be stated as follows.

Theorem 2.7 ([Nur98a]). *If $2 < p < \infty$ and $f \in L^p(\mathbb{T})$, then we have*

$$(2.13) \quad \sum_{k=1}^{\infty} k^{p-2} \left(\sup_{\substack{e \in M \\ |e| \geq k}} \frac{1}{|e|} \left| \sum_{m \in e} \widehat{f}(m) \right| \right)^p \leq C \|f\|_{L^p(\mathbb{T})}^p,$$

where M is the set of all finite arithmetic progressions in \mathbb{Z} .

We now present a (noncommutative) version of this result on the group $SU(2)$.

Theorem 2.8. *If $2 < p < \infty$ and $f \in L^p(SU(2))$, then we have*

$$(2.14) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{p-2} \left(\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \geq l}} \frac{1}{2k+1} \left| \text{Tr } \widehat{f}(k) \right| \right)^p \leq c \|f\|_{L^p(SU(2))}^p.$$

For completeness, we give a simple argument for Corollary 2.3.

Proof of Corollary 2.3. The application of the duality of L^p spaces yields

$$\|f\|_{L^p(\mathrm{SU}(2))} = \sup_{\substack{g \in L^{p'} \\ \|g\|_{L^{p'}}=1}} \left| \int_{\mathrm{SU}(2)} f(x) \overline{g(x)} dx \right|.$$

Using Plancherel's identity (2.1), we get

$$\int_{\mathrm{SU}(2)} f(x) \overline{g(x)} dx = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \mathrm{Tr} \left(\widehat{f}(l) \widehat{g}(l)^* \right).$$

It is easy to see that

$$(2l+1) = (2l+1)^{\frac{5}{2}-\frac{4}{p}+\frac{5}{2}-\frac{4}{p'}},$$

$$\left| \mathrm{Tr} \left(\widehat{f}(l) \widehat{g}(l)^* \right) \right| \leq \|\widehat{f}(l)\|_{\mathrm{HS}} \|\widehat{g}(l)\|_{\mathrm{HS}}.$$

Using these inequalities, applying Hölder inequality, for any $g \in L^{p'}$ with $\|g\|_{L^{p'}} = 1$, we have

$$\begin{aligned} \left| \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \mathrm{Tr} \left(\widehat{f}(l) \widehat{g}(l)^* \right) \right| &\leq \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}-\frac{4}{p}} \|\widehat{f}(l)\|_{\mathrm{HS}} (2l+1)^{\frac{5}{2}-\frac{4}{p'}} \|\widehat{g}(l)\|_{\mathrm{HS}} \\ &\leq \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p-4} \|\widehat{f}(l)\|_{\mathrm{HS}}^p \right)^{\frac{1}{p}} \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p'-4} \|\widehat{g}(l)\|_{\mathrm{HS}}^{p'} \right)^{\frac{1}{p'}} \\ &\leq \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p-4} \|\widehat{f}(l)\|_{\mathrm{HS}}^p \right)^{\frac{1}{p}} \|g\|_{L^{p'}}, \end{aligned}$$

where we used Theorem 2.2 in the last line. Thus, we have just proved that

$$\begin{aligned} \left| \int_{\mathrm{SU}(2)} f(x) \overline{g(x)} dx \right| &\leq \left| \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \mathrm{Tr} \left(\widehat{f}(l) \widehat{g}(l)^* \right) \right| \\ &\leq \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p-4} \|\widehat{f}(l)\|_{\mathrm{HS}}^p \right)^{\frac{1}{p}} \|g\|_{L^{p'}}. \end{aligned}$$

Taking supremum over all $g \in L^{p'}(\mathrm{SU}(2))$, we get (2.6). This proves Corollary 2.3. \square

3. LOWER BOUNDS FOR FOURIER MULTIPLIERS ON $\mathrm{SU}(2)$

Let $A: C^\infty(\mathrm{SU}(2)) \rightarrow \mathcal{D}'(\mathrm{SU}(2))$ be a continuous linear operator. Here we are concerned with left-invariant operators which means that $A \circ \tau_g = \tau_g \circ A$ for the left-translation $\tau_g f(x) = f(g^{-1}x)$. Using the Schwartz kernel theorem and the Fourier

inversion formula one can prove that the left-invariant continuous operator A can be written as a Fourier multiplier, namely, as

$$\widehat{Af}(l) = \sigma_A(l)\widehat{f}(l),$$

for the symbol $\sigma_A(l) \in \mathbb{C}^{(2l+1) \times (2l+1)}$. It follows from the Fourier inversion formula that we can write this also as

$$(3.1) \quad Af(u) = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \operatorname{Tr} \left(t^l(u) \sigma_A(l) \widehat{f}(l) \right),$$

where the symbol $\sigma_A(l)$ is given by

$$\sigma_A(l) = t^l(e)^* A t^l(e) = A t^l(e),$$

where e is an identity matrix in $SU(2)$, and $(A t^l)_{mk} = A(t_{mk}^l)$ is defined component-wise, for $-l \leq m, n \leq l$. We refer to operators in these equivalent forms as (non-commutative) Fourier multipliers. The class of these operators on $SU(2)$ and their L^p -boundedness was investigated in [CW71b, CW71a], and on general compact Lie groups in [RW13]. In particular, these authors proved Hörmander–Mikhlin type multiplier theorems in those settings, giving sufficient condition for the L^p -boundedness in terms of symbols. These conditions guarantee that the operator is of weak $(1,1)$ -type which, combined with a simple L^2 -boundedness statement, implies the boundedness on L^p for all $1 < p < \infty$.

For a general (non-invariant) operator A , its matrix symbol $\sigma_A(u, l)$ will also depend on u . Such quantization (3.1) has been consistently developed in [RT10] and [RT13]. We note that the L^p -boundedness results in [RW13] also cover such non-invariant operators.

For a noncommutative Fourier multiplier A we will write $A \in M_p^q(SU(2))$ if A extends to a bounded operator from $L^p(SU(2))$ to $L^q(SU(2))$. We introduce a norm $\|\cdot\|$ on $M_p^q(SU(2))$ by setting

$$\|A\|_{M_p^q} := \|A\|_{L^p \rightarrow L^q}.$$

Thus, we are concerned with the question of what assumptions on the symbol σ_A guarantee that $A \in M_p^q$. The sufficient conditions on σ_A for $A \in M_p^q$ were investigated in [RW13]. The aim of this section is to give a necessary condition on σ_A for $A \in M_p^q$, for $1 < p \leq 2 \leq q < \infty$.

Suppose that $1 < p \leq 2 \leq q < \infty$ and that $A: L^p(SU(2)) \rightarrow L^q(SU(2))$ is a Fourier multiplier. The Plancherel identity (2.1) implies that the operator A is bounded from $L^2(SU(2))$ to $L^2(SU(2))$ if and only if $\sup_l \|\sigma_A(l)\|_{op} < \infty$. Different other function spaces on the unitary dual have been discussed in [RT10]. Following Stein, we search for more subtle conditions on the symbols of noncommutative Fourier multipliers ensuring their $L^p - L^q$ boundedness, and we now prove a lower estimate which depends explicitly on p and q .

Theorem 3.1. *Let $1 < p \leq 2 \leq q < \infty$ and let A be a left-invariant operator on $SU(2)$ such that $A \in M_p^q(SU(2))$. Then we have*

$$(3.2) \quad \sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{\min_{n \in \{-l, \dots, +l\}} |\sigma_A(l)_{nn}|}{(2l+1)^{\frac{1}{p'} + \frac{1}{q}}} \lesssim \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))},$$

$$(3.3) \quad \sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{|\text{Tr } \sigma_A(l)|}{(2l+1)^{1 + \frac{1}{p'} + \frac{1}{q}}} \lesssim \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))}.$$

One can see a similarity between (3.2), (3.3) and (1.1) as

$$(3.4) \quad \sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{1}{(2l+1)^{\frac{1}{p'} + \frac{1}{q}}} \left(\frac{1}{2l+1} |\text{Tr } \sigma_A(l)| \right) \lesssim \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))}.$$

We also note that estimates (3.2) and (3.3) can not be immediately compared because the value of the trace in (3.3) depends on the signs of the diagonal entries of $\sigma_A(l)$.

Proof of Theorem 3.1. In [GT80] it was proven that for any $l \in \frac{1}{2}\mathbb{N}_0$ there exists a basis for $t^l \in \widehat{SU(2)}$ and a diagonal matrix coefficient t_{nn}^l (i.e. for some n , $-l \leq n \leq l$), such that

$$(3.5) \quad \|t_{nn}^l\|_{L^p(SU(2))} \cong \frac{1}{(2l+1)^{\frac{1}{p}}}.$$

Now, we use this result to establish a lower bound for the norm of $A \in M_p^q(SU(2))$. Let us fix an arbitrary $l_0 \in \frac{1}{2}\mathbb{N}_0$ and the corresponding diagonal element $t_{nn}^{l_0}$. We consider $f_{l_0}(g)$ such that its matrix-valued Fourier coefficient

$$(3.6) \quad \widehat{f_{l_0}}(l) = \text{diag}(0, \dots, 1, 0, \dots) \delta_{l_0}^l$$

has only one non-zero diagonal coefficient 1 at the n^{th} diagonal entry. Then by the Fourier inversion formula we get $f_{l_0}(g) = (2l_0+1)t_{nn}^{l_0}(g)$. By definition, we get

$$\begin{aligned} \|A\|_{L^p \rightarrow L^q} &= \sup_{f \neq 0} \frac{\|\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \text{Tr} \left(t^l(u) \sigma_A(l) \widehat{f}(l) \right)\|_{L^q(SU(2))}}{\|f\|_{L^p(SU(2))}} \\ &\geq \frac{\|\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \text{Tr} \left(t^l(u) \sigma_A(l) \widehat{f_{l_0}}(l) \right)\|_{L^q(SU(2))}}{\|f_{l_0}\|_{L^p(SU(2))}}. \end{aligned}$$

Recalling (3.6), we get

$$\|A\|_{L^p \rightarrow L^q} \gtrsim \frac{\|(2l_0+1) \text{Tr} \left(t^{l_0}(g) \sigma_A(l_0) \widehat{f_{l_0}}(l) \right)\|_{L^q(SU(2))}}{\|f_{l_0}\|_{L^p(SU(2))}}.$$

Setting $h(g) := (2l_0+1) \text{Tr} \left(t^{l_0}(g) \sigma_A(l_0) \widehat{f_{l_0}}(l_0) \right)$, we have $\widehat{h}(l) = 0$ for $l \neq l_0$, and $\widehat{h}(l_0) = \sigma_A(l_0) \widehat{f_{l_0}}(l_0)$. Consequently, we get

$$\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \geq l}} \frac{1}{2k+1} \left| \text{Tr } \widehat{h}(k) \right| = \begin{cases} 0, & l > l_0, \\ \frac{1}{2l_0+1} |\sigma_A(l_0)_{nn}|, & 1 \leq l \leq l_0. \end{cases}$$

Using this, Theorem 2.8 and (3.5), we have

$$\|A\|_{L^p \rightarrow L^q} \gtrsim \frac{\left(\sum_{l=1}^{l_0} (2l+1)^{q-2} \left(\frac{1}{2l_0+1} |\sigma_A(l_0)_{nn}| \right)^q \right)^{\frac{1}{q}}}{(2l_0+1)^{1-\frac{1}{p}}},$$

where l_0 is an arbitrary fixed half-integer. Direct calculation now shows that

$$\begin{aligned} \frac{\left(\sum_{l=1}^{l_0} (2l+1)^{q-2} \left(\frac{1}{2l_0+1} |\sigma_A(l_0)_{nn}| \right)^q \right)^{\frac{1}{q}}}{(2l_0+1)^{1-\frac{1}{p}}} &= \frac{1}{2l_0+1} |\sigma_A(l_0)_{nn}| \frac{\left(\sum_{l=1}^{l_0} (2l+1)^{q-2} \right)^{\frac{1}{q}}}{(2l_0+1)^{1-\frac{1}{p}}} \\ &= \frac{1}{2l_0+1} |\sigma_A(l_0)_{nn}| \frac{(2l_0+1)^{1-\frac{1}{q}}}{(2l_0+1)^{1-\frac{1}{p}}} \cong \frac{|\sigma_A(l_0)_{nn}|}{(2l_0+1)^{\frac{1}{p'}+\frac{1}{q}}}. \end{aligned}$$

Taking infimum over all $n \in \{-l_0, -l_0+1, \dots, l_0-1, l_0\}$ and then supremum over all half-integers, we have

$$\|A\|_{L^p \rightarrow L^q} \gtrsim \sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{\min_{n \in \{-l, \dots, +l\}} |\sigma_A(l)_{nn}|}{(2l+1)^{\frac{1}{p'}+\frac{1}{q}}}.$$

This proves estimate (3.2). Now, we will prove estimate (3.3). Let us fix some $l_0 \in \frac{1}{2}\mathbb{N}_0$ and consider now $f_{l_0}(u) := (2l_0+1)\chi_{l_0}(u)$, where $\chi_{l_0}(u) = \text{Tr } t^{l_0}(u)$ is the character of the representation t^{l_0} . Then, in particular, we have

$$(3.7) \quad \widehat{f_{l_0}}(l) = \begin{cases} I_{2l+1}, & l = l_0, \\ 0, & l \neq l_0, \end{cases}$$

where $I_{2l+1} \in \mathbb{C}^{(2l+1) \times (2l+1)}$ is the identity matrix. Using the Weyl character formula, we can write

$$\chi_{l_0}(u) = \sum_{k=-l_0}^{l_0} e^{ikt},$$

where $u = v^{-1} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} v$. The value of $\chi_{l_0}(u)$ does not depend on v since characters are central. Further, the application of the Weyl integral formula yields

$$\|f_{l_0}\|_{L^p(SU(2))} = (2l_0+1) \|\chi_{l_0}\|_{L^p(SU(2))} = (2l_0+1) \left(\int_0^{2\pi} \left| \sum_{k=-l_0}^{l_0} e^{ikt} \right|^p 2 \sin^2 t \frac{dt}{2\pi} \right)^{\frac{1}{p}}.$$

It is clear that $\left| e^{i(-l_0-1)t} \sum_{k=-l_0}^{l_0} e^{i(k+l_0+1)t} \right| = \left| \sum_{k=1}^{2l_0+1} e^{ikt} \right|$. We call $D_{2l_0+1}(t) := \sum_{k=1}^{2l_0+1} e^{ikt}$ the Dirichlet kernel. Then, we apply [Nur98a, Corollary 4] to the Dirichlet kernel $D_{2l_0+1}(t)$, to get

$$(3.8) \quad \|\chi_{l_0}\|_{L^p(SU(2))} \lesssim \|D_{2l_0+1}\|_{L^p(0,2\pi)} \cong (2l_0+1)^{1-\frac{1}{p}}.$$

By definition, we get

$$\begin{aligned} \|A\|_{L^p \rightarrow L^q} &= \sup_{f \neq 0} \frac{\left\| \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \operatorname{Tr} \left(t^l(u) \sigma_A(l) \widehat{f}(l) \right) \right\|_{L^q(\operatorname{SU}(2))}}{\|f\|_{L^p(\operatorname{SU}(2))}} \\ &\geq \frac{\left\| \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \operatorname{Tr} \left(t^l(u) \sigma_A(l) \widehat{f_{l_0}}(l) \right) \right\|_{L^q(\operatorname{SU}(2))}}{\|f_{l_0}\|_{L^p(\operatorname{SU}(2))}}. \end{aligned}$$

Recalling (3.7), we obtain

$$\|A\|_{L^p \rightarrow L^q} \gtrsim \frac{\|(2l_0+1) \operatorname{Tr} (t^{l_0}(g) \sigma_A(l_0))\|_{L^q(\operatorname{SU}(2))}}{\|f_{l_0}\|_{L^p(\operatorname{SU}(2))}}.$$

Setting $h(g) := (2l_0+1) \operatorname{Tr} (t^{l_0}(g) \sigma_A(l_0))$, we have $\widehat{h}(l) = 0$ for $l \neq l_0$, and $\widehat{h}(l_0) = \sigma_A(l_0)$. Consequently, we get

$$\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ k \geq l}} \frac{1}{2k+1} \left| \operatorname{Tr} \widehat{h}(k) \right| = \begin{cases} 0, & l > l_0, \\ \frac{1}{2l_0+1} |\operatorname{Tr} \sigma_A(l_0)|, & 1 \leq l \leq l_0. \end{cases}$$

Using this and Theorem 2.8, we have

$$\|A\|_{L^p \rightarrow L^q} \gtrsim \frac{\left(\sum_{l=1}^{l_0} (2l+1)^{q-2} \left(\frac{1}{2l_0+1} |\operatorname{Tr} \sigma_A(l_0)| \right)^q \right)^{\frac{1}{q}}}{(2l_0+1)(2l_0+1)^{1-\frac{1}{p}}},$$

where l_0 is an arbitrary fixed half-integer. Direct calculation shows that

$$\begin{aligned} \frac{\left(\sum_{l=1}^{l_0} (2l+1)^{q-2} \left(\frac{1}{2l_0+1} |\operatorname{Tr} \sigma_A(l_0)| \right)^q \right)^{\frac{1}{q}}}{(2l_0+1)(2l_0+1)^{1-\frac{1}{p}}} &= \frac{1}{2l_0+1} |\operatorname{Tr} \sigma_A(l_0)| \frac{\left(\sum_{l=1}^{l_0} (2l+1)^{q-2} \right)^{\frac{1}{q}}}{(2l_0+1)(2l_0+1)^{1-\frac{1}{p}}} \\ &= \frac{1}{2l_0+1} |\operatorname{Tr} \sigma_A(l_0)| \frac{(2l_0+1)^{1-\frac{1}{q}}}{(2l_0+1)(2l_0+1)^{1-\frac{1}{p}}} \cong \frac{|\operatorname{Tr} \sigma_A(l_0)|}{(2l_0+1)^{1+\frac{1}{p'}+\frac{1}{q}}}. \end{aligned}$$

Taking supremum over all half-integers, we have

$$\|A\|_{L^p \rightarrow L^q} \gtrsim \sup_{l \in \frac{1}{2}\mathbb{N}_0} \frac{|\operatorname{Tr} \sigma_A(l)|}{(2l+1)^{1+\frac{1}{p'}+\frac{1}{q}}}.$$

This proves the estimate (3.3) □

4. UPPER BOUNDS FOR FOURIER MULTIPLIERS ON $\operatorname{SU}(2)$

In this section we give a noncommutative $\operatorname{SU}(2)$ analogue of the upper bound for Fourier multipliers, analogous to the one on the circle \mathbb{T} in Theorem 1.1 (see also [Nur98b, NT11] for the circle case).

Theorem 4.1. *If $1 < p \leq 2 \leq q < \infty$ and A is a left-invariant operator on $SU(2)$, then we have*

$$(4.1) \quad \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))} \lesssim \sup_{s>0} s \left(\sum_{\substack{l \in \frac{1}{2}\mathbb{N}_0 \\ \|\sigma_A(l)\|_{\text{op}} > s}} (2l+1)^2 \right)^{\frac{1}{p} - \frac{1}{q}}.$$

Proof. Since A is a left-invariant operator, it acts on f via the multiplication of \widehat{f} by the symbol σ_A

$$(4.2) \quad \widehat{Af}(\pi) = \sigma_A(\pi) \widehat{f}(\pi),$$

where

$$\sigma_A(\pi) = \pi(x)^* A \pi(x) \Big|_{x=e}.$$

Let us first assume that $p \leq q'$. Since $q' \leq 2$, for $f \in C^\infty(SU(2))$ the Hausdorff-Young inequality gives

$$(4.3) \quad \begin{aligned} \|Af\|_{L^q(SU(2))} &\leq \|\widehat{Af}\|_{\ell^{q'}(\widehat{SU(2)})} = \|\sigma_A \widehat{f}\|_{\ell^{q'}(\widehat{SU(2)})} \\ &= \left(\sum_{l \in \widehat{SU(2)}} (2l+1)^{2-\frac{q'}{2}} \|\sigma_A(l) \widehat{f}(l)\|_{\text{HS}}^{q'} \right)^{\frac{1}{q'}} \\ &\leq \left(\sum_{l \in \widehat{SU(2)}} (2l+1)^{2-\frac{q'}{2}} \|\sigma_A(l)\|_{\text{op}}^{q'} \|\widehat{f}(l)\|_{\text{HS}}^{q'} \right)^{\frac{1}{q'}}. \end{aligned}$$

The case $q' \leq (p')'$ can be reduced to the case $p \leq q'$ as follows. The application of Theorem 4.2 with $G = SU(2)$ and $\mu = \{\text{Haar measure on } SU(2)\}$ yields

$$(4.4) \quad \|A\|_{L^p(SU(2)) \rightarrow L^q(SU(2))} = \|A^*\|_{L^{q'}(SU(2)) \rightarrow L^{p'}(SU(2))}.$$

The symbol $\sigma_{A^*}(l)$ of the adjoint operator A^* equals to $\sigma_A^*(l)$

$$(4.5) \quad \sigma_{A^*}(l) = \sigma_A^*(l), \quad l \in \frac{1}{2}\mathbb{N}_0,$$

and its Hilbert-Schmidt norm $\|\sigma_{A^*}(l)\|_{\text{op}}$ equals to $\|\sigma_A(l)\|_{\text{op}}$. Now, we are in a position to apply Corollary 2.6. Set $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. We observe that with $\sigma(t^l) := \|\sigma_A(t^l)\|_{\text{op}}^r I_{2l+1}$, $l \in \frac{1}{2}\mathbb{N}_0$ and $b = q'$, the assumptions of Corollary 2.6 are satisfied and we obtain

$$(4.6) \quad \begin{aligned} &\left(\sum_{l \in \widehat{SU(2)}} (2l+1)^{2-\frac{q'}{2}} \|\sigma_A(l)\|_{\text{op}}^{q'} \|\widehat{f}(l)\|_{\text{HS}}^{q'} \right)^{\frac{1}{q'}} \\ &\lesssim \left(\sup_{s>0} s \sum_{\substack{t^l \in \widehat{SU(2)} \\ \|\sigma(t^l)\|_{\text{op}}^r > s}} (2l+1)^2 \right)^{\frac{1}{r}} \|f\|_{L^p(SU(2))}, \quad f \in L^p(SU(2)), \end{aligned}$$

in view of $\frac{1}{q'} - \frac{1}{p'} = \frac{1}{p} - \frac{1}{q} = \frac{1}{r}$. Thus, for $1 < p \leq 2 \leq q < \infty$, we obtain

$$(4.7) \quad \|Af\|_{L^q(\mathrm{SU}(2))} \leq \left(\sup_{s>0} s \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{\mathrm{op}}^r > s}} (2l+1)^2 \right)^{\frac{1}{r}} \|f\|_{L^p(\mathrm{SU}(2))}.$$

Further, it can be easily checked that

$$\begin{aligned} \left(\sup_{s>0} s \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{\mathrm{op}} > s}} (2l+1)^2 \right)^{\frac{1}{r}} &= \left(\sup_{s>0} s \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma_A(t^l)\|_{\mathrm{op}} > s^{\frac{1}{r}}}} (2l+1)^2 \right)^{\frac{1}{r}} \\ &= \left(\sup_{s>0} s^r \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma_A(t^l)\|_{\mathrm{op}} > s}} (2l+1)^2 \right)^{\frac{1}{r}} = \sup_{s>0} s \left(\sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma_A(t^l)\|_{\mathrm{op}} > s}} (2l+1)^2 \right)^{\frac{1}{r}}. \end{aligned}$$

This completes the proof. \square

For the completeness, we give a short proof of Theorem 4.2 used in the proof.

Theorem 4.2. *Let (X, μ) be a measure space and $1 < p, q < \infty$. Then we have*

$$(4.8) \quad \|A\|_{L^p(X, \mu) \rightarrow L^q(X, \mu)} = \|A^*\|_{L^{q'}(X, \mu) \rightarrow L^{p'}(X, \mu)},$$

where $A^*: L^{q'}(X, \mu) \rightarrow L^{p'}(X, \mu)$ is the adjoint of A .

Proof of Theorem 4.2. Let $f \in L^p \cap L^2$ and $g \in L^{q'} \cap L^2$. By Hölder inequality, we have

$$(4.9) \quad |(Af, g)_{L^2}| = |(A^*g, f)_{L^2}| \leq \|A^*g\|_{L^{p'}} \|f\|_{L^p} \leq \|A\|_{L^{q'} \rightarrow L^{p'}} \|g\|_{L^{q'}} \|f\|_{L^p}.$$

Thus, we get

$$(4.10) \quad \|A\|_{L^p \rightarrow L^q} \leq \|A^*\|_{L^{q'} \rightarrow L^{p'}}.$$

Analogously, we show that

$$(4.11) \quad \|A^*\|_{L^{q'} \rightarrow L^{p'}} \leq \|A\|_{L^p \rightarrow L^q}.$$

The combination of (4.10) and (4.11) yields

$$\|A\|_{L^p \rightarrow L^q} = \|A^*\|_{L^{q'} \rightarrow L^{p'}}.$$

This completes the proof. \square

5. PROOFS OF THEOREMS FROM SECTION 2

Proof of Theorem 2.4. Let μ give measure $\|\sigma(t^l)\|_{op}^2(2l+1)^2, l \in \frac{1}{2}\mathbb{N}_0$ to the set consisting of the single point $\{t^l\}, t^l \in \widehat{SU(2)}$, and measure zero to a set which does not contain any of these points, i.e.

$$\mu\{t^l\} := \|\sigma(t^l)\|_{op}^2(2l+1)^2.$$

We define the space $L^p(\widehat{SU(2)}, \mu)$, $1 \leq p < \infty$, as the space of complex (or real) sequences $a = \{a_l\}_{l \in \frac{1}{2}\mathbb{N}_0}$ such that

$$(5.1) \quad \|a\|_{L^p(\widehat{SU(2)}, \mu)} := \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} |a_l|^p \|\sigma(t^l)\|_{op}^2(2l+1)^2 \right)^{\frac{1}{p}} < \infty.$$

We will show that the sub-linear operator

$$A: L^p(SU(2)) \ni f \mapsto Af = \left\{ \frac{\|\widehat{f}(t^l)\|_{HS}}{\sqrt{2l+1}\|\sigma(t^l)\|_{op}} \right\}_{t^l \in \widehat{SU(2)}} \in L^p(\widehat{SU(2)}, \mu)$$

is well-defined and bounded from $L^p(SU(2))$ to $L^p(\widehat{SU(2)}, \mu)$ for $1 < p \leq 2$. In other words, we claim that we have the estimate

$$(5.2) \quad \|Af\|_{L^p(\widehat{SU(2)}, \mu)} = \left(\sum_{t^l \in \widehat{SU(2)}} \left(\frac{\|\widehat{f}(t^l)\|_{HS}}{\sqrt{2l+1}\|\sigma(t^l)\|_{op}} \right)^p \|\sigma(t^l)\|_{op}^2(2l+1)^2 \right)^{\frac{1}{p}} \\ \lesssim K_{\sigma}^{\frac{2-p}{p}} \|f\|_{L^p(SU(2))},$$

which would give (2.8) and where we set $K_{\sigma} := \sup_{s>0} s \sum_{\substack{t^l \in \widehat{SU(2)} \\ \|\sigma(t^l)\|_{op} \geq s}} (2l+1)^2$. We

will show that A is of weak type (2,2) and of weak-type (1,1). For definition and discussions we refer to Section 6 where we give definitions of weak-type, formulate and prove Marcinkiewicz interpolation Theorem 6.1. More precisely, with the distribution function ν as in Theorem 6.1, we show that

$$(5.3) \quad \nu_{\widehat{SU(2)}}(y; Af) \leq \left(\frac{M_2 \|f\|_{L^2(SU(2))}}{y} \right)^2 \quad \text{with norm } M_2 = 1,$$

$$(5.4) \quad \nu_{\widehat{SU(2)}}(y; Af) \leq \frac{M_1 \|f\|_{L^1(SU(2))}}{y} \quad \text{with norm } M_1 = K_{\sigma}.$$

Then (5.2) would follow by Marcinkiewicz interpolation Theorem 6.1. Now, to show (5.3), using Plancherel's identity (2.1), we get

$$\begin{aligned} y^2 \nu_{\widehat{\mathrm{SU}(2)}}(y; Af) &\leq \|Af\|_{L^p(\widehat{\mathrm{SU}(2)}, \mu)}^2 := \sum_{t^l \in \widehat{\mathrm{SU}(2)}} \left(\frac{\|\widehat{f}(t^l)\|_{\mathrm{HS}}}{\sqrt{2l+1} \|\sigma(t^l)\|_{op}} \right)^2 \|\sigma(t^l)\|_{op}^2 (2l+1)^2 \\ &= \sum_{t^l \in \widehat{\mathrm{SU}(2)}} (2l+1) \|\widehat{f}(t^l)\|_{\mathrm{HS}}^2 = \|\widehat{f}\|_{\ell^2(\widehat{\mathrm{SU}(2)})}^2 = \|f\|_{L^2(\mathrm{SU}(2))}^2. \end{aligned}$$

Thus, A is of type (2,2) with norm $M_2 \leq 1$. Further, we show that A is of weak-type (1,1) with norm $M_1 = C$; more precisely, we show that

$$(5.5) \quad \nu_{\widehat{\mathrm{SU}(2)}}\{t^l \in \widehat{\mathrm{SU}(2)} : \frac{\|\widehat{f}(t^l)\|_{\mathrm{HS}}}{\sqrt{2l+1} \|\sigma(t^l)\|_{op}} > y\} \lesssim K_\sigma \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{y}.$$

The left-hand side here is the weighted sum $\sum \|\sigma(t^l)\|_{op}^2 (2l+1)^2$ taken over those $t^l \in \widehat{\mathrm{SU}(2)}$ for which $\frac{\|\widehat{f}(t^l)\|_{\mathrm{HS}}}{\sqrt{2l+1} \|\sigma(t^l)\|_{op}} > y$. From the definition of the Fourier transform it follows that

$$\|\widehat{f}(t^l)\|_{\mathrm{HS}} \leq \sqrt{2l+1} \|f\|_{L^1(\mathrm{SU}(2))}.$$

Therefore, we have

$$y < \frac{\|\widehat{f}(t^l)\|_{\mathrm{HS}}}{\sqrt{2l+1} \|\sigma(t^l)\|_{op}} \leq \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{\|\sigma(t^l)\|_{op}}.$$

Using this, we get

$$\left\{ t^l \in \widehat{\mathrm{SU}(2)} : \frac{\|\widehat{f}(t^l)\|_{\mathrm{HS}}}{\sqrt{2l+1} \|\sigma(t^l)\|_{op}} > y \right\} \subset \left\{ t^l \in \widehat{\mathrm{SU}(2)} : \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{\|\sigma(t^l)\|_{op}} > y \right\}$$

for any $y > 0$. Consequently,

$$\mu \left\{ t^l \in \widehat{\mathrm{SU}(2)} : \frac{\|\widehat{f}(t^l)\|_{\mathrm{HS}}}{\sqrt{2l+1} \|\sigma(t^l)\|_{op}} > y \right\} \leq \mu \left\{ t^l \in \widehat{\mathrm{SU}(2)} : \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{\|\sigma(t^l)\|_{op}} > y \right\}.$$

Setting $v := \frac{\|f\|_{L^1(\mathrm{SU}(2))}}{y}$, we get

$$(5.6) \quad \mu \left\{ t^l \in \widehat{\mathrm{SU}(2)} : \frac{\|\widehat{f}(t^l)\|_{\mathrm{HS}}}{\sqrt{2l+1} \|\sigma(t^l)\|_{op}} > y \right\} \leq \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{op} \leq v}} \|\sigma(t^l)\|_{op}^2 (2l+1)^2.$$

We claim that

$$(5.7) \quad \sum_{\substack{t^l \in \widehat{\mathrm{SU}(2)} \\ \|\sigma(t^l)\|_{op} \leq v}} \|\sigma(t^l)\|_{op}^2 (2l+1)^2 \lesssim K_\sigma v.$$

In fact, we have

$$\sum_{\substack{t^l \in \widehat{SU(2)} \\ \|\sigma(t^l)\|_{op} \leq v}} \|\sigma(t^l)\|_{op}^2 (2l+1)^2 = \sum_{\substack{t^l \in \widehat{SU(2)} \\ \|\sigma(t^l)\|_{op} \leq v}} (2l+1)^2 \int_0^{\|\sigma(t^l)\|_{op}^2} d\tau.$$

We can interchange sum and integration to get

$$\sum_{\substack{t^l \in \widehat{SU(2)} \\ \|\sigma(t^l)\|_{op} \leq v}} (2l+1)^2 \int_0^{\|\sigma(t^l)\|_{op}^2} d\tau = \int_0^v d\tau \sum_{\substack{t^l \in \widehat{SU(2)} \\ \tau^{\frac{1}{2}} \leq \|\sigma(t^l)\|_{op} \leq v}} (2l+1)^2.$$

Further, we make a substitution $\tau = s^2$, yielding

$$\begin{aligned} \int_0^{v^2} d\tau \sum_{\substack{t^l \in \widehat{SU(2)} \\ \tau^{\frac{1}{2}} \leq \|\sigma(t^l)\|_{op} \leq v}} (2l+1)^2 &= 2 \int_0^v s ds \sum_{\substack{t^l \in \widehat{SU(2)} \\ s \leq \|\sigma(t^l)\|_{op} \leq v}} (2l+1)^2 \\ &\leq 2 \int_0^v s ds \sum_{\substack{t^l \in \widehat{SU(2)} \\ s \leq \|\sigma(t^l)\|_{op}}} (2l+1)^2. \end{aligned}$$

Since

$$s \sum_{\substack{t^l \in \widehat{SU(2)} \\ s \leq \|\sigma(t^l)\|_{op}}} (2l+1)^2 \leq \sup_{s>0} s \sum_{\substack{t^l \in \widehat{SU(2)} \\ s \leq \|\sigma(t^l)\|_{op}}} (2l+1)^2 =: K_\sigma$$

is finite by the definition of K_σ , we have

$$2 \int_0^v s ds \sum_{\substack{t^l \in \widehat{SU(2)} \\ s \leq \|\sigma(t^l)\|_{op}}} (2l+1)^2 \lesssim K_\sigma v.$$

This proves (5.7). We have just proved inequalities (5.3), (5.4). Then by using Marcinkiewicz' interpolation theorem (Theorem 6.1 from Section 6) with $p_1 = 1$, $p_2 = 2$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{2}$ we now obtain

$$\begin{aligned} \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} \left(\frac{\|\widehat{f}(\pi)\|_{\text{HS}}}{\sqrt{2l+1} \|\sigma(\pi)\|_{op}} \right)^p \|\sigma(\pi)\|_{op}^2 (2l+1)^2 \right)^{\frac{1}{p}} \\ = \|Af\|_{L^p(\widehat{SU(2)}, \mu)} \lesssim K_\sigma^{\frac{2-p}{p}} \|f\|_{L^p(SU(2))}. \end{aligned}$$

This completes the proof. □

Now we prove the Hardy–Littlewood type inequality given in Theorem 2.2.

Proof of Theorem 2.2. Let ν give measure $\frac{1}{(2l+1)^4}$ to the set consisting of the single point $l, l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$, and measure zero to a set which does not contain any of these points. We will show that the sub-linear operator

$$Tf := \{(2l+1)^{\frac{5}{2}} \|\widehat{f}(l)\|_{\text{HS}}\}_{l \in \frac{1}{2}\mathbb{N}_0}$$

is well-defined and bounded from $L^p(\text{SU}(2))$ to $L^p(\frac{1}{2}\mathbb{N}_0, \nu)$ for $1 < p \leq 2$, with

$$\|Tf\|_{L^p(\widehat{\text{SU}(2)}, \nu)} = \left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} \left((2l+1)^{\frac{5}{2}} \|\widehat{f}(l)\|_{\text{HS}} \right)^p \cdot (2l+1)^{-4} \right)^{\frac{1}{p}}.$$

This will prove Theorem 2.2. We first show that T is of type $(2, 2)$ and weak type $(1, 1)$. Using Plancherel's identity (2.1), we get

$$\begin{aligned} \|Tf\|_{L^p(\widehat{\text{SU}(2)}, \nu)}^2 &= \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5p}{2}-4} \|\widehat{f}(l)\|_{\text{HS}}^2 = \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1) \|\widehat{f}(l)\|_{\text{HS}}^2 \\ &= \|\widehat{f}\|_{\ell^2(\widehat{\text{SU}(2)})}^2 = \|f\|_{L^2(\text{SU}(2))}^2. \end{aligned}$$

Thus, T is of type $(2, 2)$.

Further, we show that T is of *weak type* $(1, 1)$; more precisely we show that

$$(5.8) \quad \nu\{l \in \frac{1}{2}\mathbb{N}_0 : (2l+1)^{\frac{5}{2}} \|\widehat{f}(l)\|_{\text{HS}} > y\} \leq \frac{4}{3} \frac{\|f\|_{L^1(\text{SU}(2))}}{y}.$$

The left-hand side here is the sum $\sum \frac{1}{(2l+1)^4}$ taken over those $l \in \frac{1}{2}\mathbb{N}_0$ for which $(2l+1)^{\frac{5}{2}} \|\widehat{f}(l)\|_{\text{HS}} > y$. From the definition of the Fourier transform it follows that

$$\|\widehat{f}(l)\|_{\text{HS}} \leq \sqrt{2l+1} \|f\|_{L^1(\text{SU}(2))}.$$

Therefore, we have

$$y < (2l+1)^{\frac{5}{2}} \|\widehat{f}(l)\|_{\text{HS}} \leq (2l+1)^{\frac{5}{2}+\frac{1}{2}} \|f\|_{L^1(\text{SU}(2))}.$$

Using this, we get

$$\left\{ l \in \frac{1}{2}\mathbb{N}_0 : (2l+1)^{\frac{5}{2}} \|\widehat{f}(l)\|_{\text{HS}} > y \right\} \subset \left\{ l \in \frac{1}{2}\mathbb{N}_0 : (2l+1) > \left(\frac{y}{\|f\|_{L^1}} \right)^{\frac{1}{3}} \right\}$$

for any $y > 0$. Consequently,

$$\nu \left\{ l \in \frac{1}{2}\mathbb{N}_0 : (2l+1)^{\frac{5}{2}} \|\widehat{f}(l)\|_{\text{HS}} > y \right\} \leq \nu \left\{ l \in \frac{1}{2}\mathbb{N}_0 : (2l+1) > \left(\frac{y}{\|f\|_{L^1}} \right)^{\frac{1}{3}} \right\}.$$

We set $w := \left(\frac{y}{\|f\|_{L^1(\text{SU}(2))}} \right)^{\frac{1}{3}}$. Now, we estimate $\nu \{l \in \frac{1}{2}\mathbb{N}_0 : (2l+1) > w\}$. By definition, we have

$$\nu \left\{ l \in \frac{1}{2}\mathbb{N}_0 : (2l+1) > \left(\frac{y}{\|f\|_{L^1}} \right)^{\frac{1}{3}} \right\} = \sum_{n > w}^{\infty} \frac{1}{n^4}.$$

In order to estimate this series, we introduce the following lemma.

Lemma 5.1. *Suppose $\beta > 1$ and $w > 0$. Then we have*

$$(5.9) \quad \sum_{n>w}^{\infty} \frac{1}{n^{\beta}} \leq \begin{cases} \frac{\beta}{\beta-1}, & w \leq 1, \\ \frac{1}{\beta-1} \frac{1}{w^{\beta-1}}, & w > 1. \end{cases}$$

The proof is rather straightforward. Now, suppose $w \leq 1$. Then applying this lemma with $\beta = 4$, we have

$$\sum_{n>w}^{\infty} \frac{1}{n^4} \leq \frac{4}{3}.$$

Since $1 \leq \frac{1}{w^3}$, we obtain

$$\sum_{n>w}^{\infty} \frac{1}{n^4} \leq \frac{4}{3} \leq \frac{4}{3} \frac{1}{w^3}.$$

Recalling that $w = \left(\frac{y}{\|f\|_{L^1(SU(2))}} \right)^{\frac{1}{3}}$, we finally obtain

$$\nu \left\{ l \in \frac{1}{2}\mathbb{N}_0 : (2l+1) > \left(\frac{y}{\|f\|_{L^1}} \right)^{\frac{1}{3}} \right\} = \sum_{n>w}^{\infty} \frac{1}{n^4} \leq \frac{4}{3} \frac{\|f\|_{L^1(SU(2))}}{y}.$$

Now, if $w > 1$, then we have

$$\sum_{n>w}^{\infty} \frac{1}{n^4} \leq \frac{1}{3} \frac{1}{w^3} = \frac{4}{3} \frac{\|f\|_{L^1}}{y}.$$

Finally, we get

$$\nu \left\{ l \in \frac{1}{2}\mathbb{N}_0 : (2l+1) > \left(\frac{y}{\|f\|_{L^1}} \right)^{\frac{1}{3}} \right\} \leq \frac{4}{3} \frac{\|f\|_{L^1(SU(2))}}{y}.$$

This proves (5.8).

By Marcinkiewicz interpolation Theorem 6.1 with $p_1 = 1, p_2 = 2$, we obtain

$$\left(\sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{\frac{5}{2}p-4} \|\widehat{f}(l)\|_{\mathbb{H}^p}^p \right)^{\frac{1}{p}} = \|Tf\|_{L^p(\widehat{SU(2)}, \nu)} \leq c_p \|f\|_{L^p(SU(2))}.$$

This completes the proof of Theorem 2.2. \square

Now we prove Theorem 2.8.

Proof of Theorem 2.8. We first simplify the expression for $\text{Tr } \widehat{f}(k)$. By definition, we have

$$\widehat{f}(k) = \int_{SU(2)} f(u) T^k(u)^* du, \quad k \in \frac{1}{2}\mathbb{N}_0,$$

where T^k is a finite-dimensional representation of $\widehat{SU(2)}$ as in Section 2. Using this, we get

$$(5.10) \quad \text{Tr } \widehat{f}(k) = \int_{SU(2)} f(u) \overline{\chi_k(u)} du,$$

where $\chi_k(u) = \text{Tr } T^k(u)$, $k \in \frac{1}{2}\mathbb{N}_0$, where we changed the notation from t^k to T^k to avoid confusing with the notation that follows. The characters $\chi_k(u)$ are constant on the conjugacy classes of $\text{SU}(2)$ and we follow [Vil68] to describe these classes explicitly.

It is well known from linear algebra that any unitary unimodular matrix u can be written in the form $u = u_1 \delta u_1^{-1}$, where $u_1 \in \text{SU}(2)$ and δ is a diagonal matrix of the form

$$(5.11) \quad \delta = \begin{pmatrix} e^{\frac{it}{2}} & 0 \\ 0 & e^{-\frac{it}{2}} \end{pmatrix},$$

where $\lambda = e^{\frac{it}{2}}$ and $\frac{1}{\lambda} = e^{-\frac{it}{2}}$ are the eigenvalues of u . Moreover, among the matrices equivalent to u there is only one other diagonal matrix, namely, the matrix δ' obtained from δ by interchanging the diagonal elements.

Hence, classes of conjugate elements in $\text{SU}(2)$ are given by one parameter t , varying in the limits $-2\pi \leq t \leq 2\pi$, where the parameters t and $-t$ give one and the same class. Therefore, we can regard the characters $\chi_k(u)$ as functions of one variable t , which ranges from 0 to 2π .

The special unitary group $\text{SU}(2)$ is isomorphic to the group of unit quaternions. Hence, the parameter t has a simple geometrical meaning - it is equal to angle of rotation which corresponds to the matrix u .

Let us now derive an explicit expression for the $\chi_k(u)$ as function of t . It was shown e.g. in [RT10] that $T^k(\delta)$ is a diagonal matrix with the numbers e^{-int} , $-k \leq n \leq k$ on its principal diagonal.

Let $u = u_1 \delta u_1^{-1}$. Since characters are constant on conjugacy classes of elements, we get

$$(5.12) \quad \chi_k(u) = \chi_k(\delta) = \text{Tr } (T^k(\delta)) = \sum_{n=-k}^k e^{int}.$$

It is natural to express the invariant integral over $\text{SU}(2)$ in (5.10) in new parameters, one of which is t .

Since special unitary group $\text{SU}(2)$ is diffeomorphic to the unit sphere \mathbb{S}^3 in \mathbb{R}^4 (see, e.g., [RT10]), with

$$\text{SU}(2) \ni u = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \longleftrightarrow \varphi(u) = x = (x_1, x_2, x_3, x_4) \in \mathbb{S}^3,$$

we have

$$(5.13) \quad \int_{\text{SU}(2)} f(u) \chi_k(u) du = \int_{\mathbb{S}^3} f(x) \chi_k(x) dS,$$

where $f(x) := f(\varphi^{-1}(x))$, and $\chi_k(x) := \chi_k(\varphi^{-1}(x))$. In order to find an explicit formula for this integral over \mathbb{S}^3 , we consider the parametrisation

$$\begin{aligned} x_1 &= \cos \frac{t}{2}, \\ x_2 &= v, \\ x_3 &= \sqrt{\sin^2 \frac{t}{2} - v^2} \cdot \cos h, \\ x_4 &= \sqrt{\sin^2 \frac{t}{2} - v^2} \cdot \sin h, \quad (t, v, h) \in D, \end{aligned}$$

where $D = \{(t, v, h) \in \mathbb{R}^3 : |v| \leq \sin \frac{t}{2}, 0 \leq t, h \leq 2\pi\}$.

The reader will have no difficulty in showing that $dS = \sin \frac{t}{2} dt dv dh$. Therefore, we have

$$\int_{\mathbb{S}^3} f(x) \chi_k(t) dS = \int_D f(h, v, t) \chi_k(t) \sin \frac{t}{2} dh dv dt.$$

Combining this and (5.13), we get

$$\text{Tr } \widehat{f}(k) = \int_D f(h, v, t) \chi_k(t) \sin \frac{t}{2} dh dv dt.$$

Thus, we have expressed the invariant integral over $SU(2)$ in the parameters t, v, h . The application of Fubini's Theorem yields

$$\int_D f(h, v, t) \chi_k(t) \sin \frac{t}{2} dh dv dt = \int_0^{2\pi} \chi_k(t) \sin \frac{t}{2} dt \int_{-\sin \frac{t}{2}}^{\sin \frac{t}{2}} dv \int_0^{2\pi} f(h, v, t) dh.$$

Combining this and (5.12), we obtain

$$\text{Tr } \widehat{f}(k) = \int_0^{2\pi} dt \sum_{n=-k}^k e^{int} \sin \frac{t}{2} \int_{-\sin \frac{t}{2}}^{\sin \frac{t}{2}} dv \int_0^{2\pi} f(h, v, t) dh.$$

Interchanging summation and integration, we get

$$\text{Tr } \widehat{f}(k) = \sum_{n=-k}^k \int_0^{2\pi} e^{int} \sin \frac{t}{2} dt \int_{-\sin \frac{t}{2}}^{\sin \frac{t}{2}} dv \int_0^{2\pi} f(h, v, t) dh.$$

By making the change of variables $t \rightarrow 2t$, we get

$$(5.14) \quad \text{Tr } \widehat{f}(k) = \sum_{n=-k}^k \int_0^{\pi} e^{-i2nt} \cdot 2 \sin t dt \int_{-\sin t}^{\sin t} dv \int_0^{2\pi} f(h, v, 2t) dh.$$

Let us now apply Theorem 2.7 in $L^p(\mathbb{T})$. To do this we introduce some notation. Denote

$$F(t) := 2 \sin t \int_{-\sin t}^{\sin t} \int_0^{2\pi} f(h, v, 2t) dh dv, \quad t \in (0, \pi).$$

We extend $F(t)$ periodically to $[0, 2\pi)$, that is $F(x + \pi) = F(x)$. Since $f(t, v, h)$ is integrable, the integrability of $F(t)$ follows immediately from Fubini's Theorem. Thus function $F(t)$ has a Fourier series representation

$$F(t) \sim \sum_{k \in \mathbb{Z}} \widehat{F}(k) e^{ikt},$$

where the Fourier coefficients are computed by

$$\widehat{F}(k) = \frac{1}{2\pi} \int_{[0, 2\pi]} F(t) e^{-ikt} dt.$$

Let A_k be a $2k + 1$ -element arithmetic sequence with step 2 and initial term $-2k$, i.e.,

$$A_k = \{-2k, -2k + 2, \dots, 2k\} = \{-2k + 2j\}_{j=0}^{2k}.$$

Using this notation and (5.14), we have

$$(5.15) \quad \text{Tr } \widehat{f}(k) = \sum_{n \in A_k} \widehat{F}(n).$$

Define

$$B = \{A_k\}_{k=1}^{\infty}.$$

Using the fact that B is a subset of the set M of all finite arithmetic progressions, and (5.15), we have

$$(5.16) \quad \sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ 2k+1 \geq 2l+1}} \frac{1}{2k+1} \left| \text{Tr } \widehat{f}(k) \right| \leq \sup_{\substack{e \in B \\ |e| \geq 2l+1}} \frac{1}{|e|} \left| \sum_{i \in e} \widehat{F}(i) \right| \leq \sup_{\substack{e \in M \\ |e| \geq 2l+1}} \frac{1}{|e|} \left| \sum_{i \in e} \widehat{F}(i) \right|.$$

Denote $m := 2l + 1$. If l runs over $\frac{1}{2}\mathbb{N}_0$, then m runs over \mathbb{N} . Using (5.16), we get

$$(5.17) \quad \sum_{l \in \frac{1}{2}\mathbb{N}_0} (2l+1)^{p-2} \left(\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ 2k+1 \geq 2l+1}} \frac{1}{2k+1} \left| \text{Tr } \widehat{f}(k) \right| \right)^p \leq \sum_{m \in \mathbb{N}} m^{p-2} \left(\sup_{\substack{e \in M \\ |e| \geq m}} \frac{1}{|e|} \left| \sum_{i \in e} \widehat{F}(i) \right| \right)^p.$$

Application of inequality (2.13) yields

$$(5.18) \quad \sum_{m \in \mathbb{N}} m^{p-2} \left(\sup_{\substack{e \in M \\ |e| \geq m}} \frac{1}{|e|} \left| \sum_{i \in e} \widehat{F}(i) \right| \right)^p \leq c \|F\|_{L^p(0, 2\pi)}^p.$$

Using Hölder inequality, we obtain

$$\int_0^\pi |F(t)|^p dt \lesssim \int_0^\pi \sin t dt \int_{-\sin t}^{\sin t} dv \int_0^{2\pi} |f(h, v, 2t)|^p dh.$$

By making the change of variables $t \rightarrow \frac{t}{2}$ in the right hand side integral, we get

$$\int_0^\pi |F(t)|^p dt \lesssim \left(\int_0^{2\pi} \sin \frac{t}{2} dt \int_{-\sin \frac{t}{2}}^{\sin \frac{t}{2}} dv \int_0^{2\pi} |f(h, v, t)|^p dh \right)^{\frac{1}{p}}.$$

Thus, we have proved that

$$(5.19) \quad \|F\|_{L^p(0, \pi)} \leq c_p \|f\|_{L^p(SU(2))},$$

where c_p depending only on p . Combining (5.16), (5.17) and (5.19), we obtain

$$\sum_{m \in \mathbb{N}} m^{p-2} \left(\sup_{\substack{k \in \frac{1}{2}\mathbb{N}_0 \\ 2k+1 \geq m}} \frac{1}{2k+1} |\text{Tr } \widehat{f}(k)| \right)^p \leq c \|f\|_{L^p(SU(2))}^p.$$

This completes the proof. \square

6. MARCINKIEWICZ INTERPOLATION THEOREM

In this section we formulate and prove Marcinkiewicz interpolation theorem for linear mappings between G and the space of matrix-valued sequences Σ that will be realised via

$$\Sigma := \{h = \{h(\pi)\}_{\pi \in \widehat{G}}, h(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}\}.$$

Thus, a linear mapping $A: \mathcal{D}'(G) \rightarrow \Sigma$ takes a function to a matrix valued sequence, i.e.

$$f \mapsto Af =: h = \{h(\pi)\}_{\pi \in \widehat{G}},$$

where

$$h(\pi) \in \mathbb{C}^{d_\pi \times d_\pi}, \pi \in \widehat{G}.$$

We say that a linear operator A is of strong type (p, q) , if for every $f \in L^p(G)$, we have $Af \in \ell^q(\widehat{G}, \Sigma)$ and

$$\|Af\|_{\ell^q(\widehat{G}, \Sigma)} \leq M \|f\|_{L^p(G)},$$

where M is independent of f , and the space $\ell^q(\widehat{G}, \Sigma)$ defined by the norm

$$(6.1) \quad \|h\|_{\ell^q(\widehat{G}, \Sigma)} := \left(\sum_{\pi \in \widehat{G}} d^{p(\frac{2}{p}-\frac{1}{2})} \|h(\pi)\|_{HS}^p \right)^{\frac{1}{p}}$$

(2.9). The least M for which this is satisfied is taken to be the strong (p, q) -norm of the operator A .

Denote the distribution functions of f and h by $\mu_G(t; f)$ and $\nu_{\widehat{G}}(u; h)$, respectively, i.e.

$$(6.2) \quad \mu_G(x; f) := \int_{\substack{u \in G \\ |f(u)| \geq x}} du, \quad x > 0,$$

$$(6.3) \quad \nu_{\widehat{G}}(y; h) := \sum_{\substack{\pi \in \widehat{G} \\ \frac{\|h(\pi)\|_{\text{HS}}}{\sqrt{d_\pi}} \geq y}} d_\pi^2, \quad y > 0.$$

Then

$$\begin{aligned} \|f\|_{L^p(G)}^p &= \int_G |f(u)|^p du = p \int_0^{+\infty} x^{p-1} \mu_G(x; f) dx, \\ \|h\|_{\ell^q(\widehat{G}, \Sigma)}^q &= \sum_{\pi \in \widehat{G}} d_\pi^2 \left(\frac{\|h(\pi)\|_{\text{HS}}}{\sqrt{d_\pi}} \right)^q = q \int_0^{+\infty} u^{q-1} \nu_{\widehat{G}}(y; h) dy. \end{aligned}$$

A linear operator $A: \mathcal{D}'(\text{SU}(2)) \rightarrow \Sigma$ satisfying

$$(6.4) \quad \nu_{\widehat{G}}(y; Af) \leq \left(\frac{M}{y} \|f\|_{L^p(G)} \right)^q$$

is said to be of *weak type* (p, q) ; the least value of M in (6.4) is called weak (p, q) norm of A .

Every operation of strong type (p, q) is also of weak type (p, q) , since

$$y (\nu_{\widehat{G}}(y; Af))^{\frac{1}{q}} \leq \|Af\|_{L^q(\widehat{G})} \leq M \|f\|_{L^p(G)}.$$

Theorem 6.1. *Let $1 \leq p_1 < p < p_2 < \infty$. Suppose that a linear operator A from $\mathcal{D}'(G)$ to Σ is simultaneously of weak types (p_1, p_1) and (p_2, p_2) , with norms M_1 and M_2 , respectively, i.e.*

$$(6.5) \quad \nu_{\widehat{G}}(y; Af) \leq \left(\frac{M_1}{y} \|f\|_{L^{p_1}(G)} \right)^{p_1},$$

$$(6.6) \quad \nu_{\widehat{G}}(y; Af) \leq \left(\frac{M_2}{y} \|f\|_{L^{p_2}(G)} \right)^{p_2}.$$

Then for any $p \in (p_1, p_2)$ the operator A is of strong type (p, p) and we have

$$(6.7) \quad \|Af\|_{\ell^p(\widehat{G}, \Sigma)} \leq M_1^{1-\theta} M_2^\theta \|f\|_{L^p(G)}, \quad 0 < \theta < 1,$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

The proof is done in analogy to Zygmund [Zyg56] adapting it to our setting.

Proof. Let $f \in L^p(G)$. We have to prove inequality (6.7). By definition, we have

$$(6.8) \quad \|Af\|_{\ell^p(\widehat{G}, \Sigma)}^p = \sum_{\pi \in \widehat{G}} d_\pi^2 \left(\frac{\|Af(\pi)\|_{\text{HS}}}{\sqrt{d_\pi}} \right)^p = \int_0^{+\infty} p x^{p-1} \nu_{\widehat{G}}(x; Af) dx.$$

For a fixed arbitrary $z > 0$ we consider the decomposition

$$f = f_1 + f_2,$$

where $f_1 = f$ whenever $|f| < z$, and $f_1 = 0$ otherwise; thus $|f_2| > z$ or else $f_2 = 0$. Since $f \in L^p(G)$ the same holds for f_1 and f_2 ; it follows that f_1 is in $L^{p_1}(G)$ and $f_2 \in L^{p_2}(G)$. Hence Af_1 and Af_2 exist, by hypothesis, and so does $Af = A(f_1 + f_2)$. It follows that

$$(6.9) \quad |f_1| = \min(|f|, z), \quad |f| = |f_1| + |f_2|.$$

The inequality

$$\|A(f_1 + f_2)(\pi)\|_{\text{HS}} \leq \|Af_1(\pi)\|_{\text{HS}} + \|Af_2(\pi)\|_{\text{HS}}, \quad \pi \in \widehat{G}$$

leads to an inclusion

$$\begin{aligned} \left\{ \pi \in \widehat{G} : \frac{\|Af(\pi)\|_{\text{HS}}}{\sqrt{d_\pi}} \geq y \right\} &\subset \\ &\subset \left\{ \pi \in \widehat{G} : \frac{\|Af_1(\pi)\|_{\text{HS}}}{\sqrt{d_\pi}} \geq \frac{y}{2} \right\} \cup \left\{ \pi \in \widehat{G} : \frac{\|Af_2(\pi)\|_{\text{HS}}}{\sqrt{d_\pi}} \geq \frac{y}{2} \right\}. \end{aligned}$$

Then applying assumptions (6.5) and (6.6) to f_1 and f_2 , we obtain

$$\begin{aligned} (6.10) \quad \nu_{\widehat{G}}(y; Af) &\leq \nu_{\widehat{G}}\left(\frac{y}{2}; Af_1\right) + \nu_{\widehat{G}}\left(\frac{y}{2}; Af_2\right) \\ &\leq M_1^{p_1} y^{-p_1} \|f_1\|_{L^{p_1}(G)}^{p_1} + M_1^{p_2} y^{-p_2} \|f_2\|_{L^{p_2}(G)}^{p_2}. \end{aligned}$$

The right side here depends on z and the main idea of the proof consists in defining z as a suitable monotone functions of t , $z = z(t)$, to be determined later. By (6.9)

$$\begin{aligned} \mu_G(t; f_1) &= \mu_G(t; f), \quad \text{for } 0 < t \leq z, \\ \mu_G(t; f_1) &= 0, \quad \text{for } t > z, \\ \mu_G(t; f_2) &= \mu_G(t + z; f), \quad \text{for } t > 0. \end{aligned}$$

Here, the last equation is a consequence of the fact that wherever $f_2 \neq 0$ we must have $|f_1| = z$, and so the second equation (6.9) takes the form $|f| = z + |f_2|$.

It follows from (6.10) that the last integral in (6.8) is less than

$$\begin{aligned} (6.11) \quad M_1^{p_1} \int_0^{+\infty} y^{p-p_1-1} \left\{ \int_G |f_1(u)|^{p_1} du \right\}^{\frac{p_1}{p_1}} dy &+ \\ &+ M_2^{p_2} \int_0^{+\infty} y^{p-p_2-1} \left\{ \int_G |f_2(u)|^{p_2} du \right\}^{\frac{p_2}{p_2}} dy \\ &= M_1^{p_1} p_1 \int_0^{+\infty} y^{p-p_1-1} \left\{ \int_0^z x^{p_1-1} \mu_G(x; f) dx \right\} dt \\ &\quad + M_2^{p_2} p_2 \int_0^{+\infty} y^{p-p_2-1} \left\{ \int_z^{+\infty} (x-z)^{p_2-1} \mu_G(x; f) dx \right\} dt. \end{aligned}$$

Set $z(y) = \frac{A}{y}$. Denote by I_1 and I_2 the two double integrals last written. We change the order of integration in I_1

$$\begin{aligned}
 (6.12) \quad I_1 &= \int_0^{+\infty} t^{p-p_1-1} \left\{ \int_0^z u^{p_1-1} \mu_G(u; f) du \right\} dt \\
 &= \int_0^{+\infty} x^{p_1-1} \mu_G(x; f) \left\{ \int_0^{Ax} y^{p-p_1-1} dy \right\} dx \\
 &= \frac{A^{p-p_1}}{p-p_1} \int_0^{+\infty} x^{p_1-1+p-p_1} \mu_G(x; f) dx.
 \end{aligned}$$

Similarly, making a substitution $x - z \rightarrow x$ and using (6.9) we see that I_2 is

$$\begin{aligned}
 (6.13) \quad I_2 &= M_2^{p_2} p_2 \int_0^{+\infty} y^{p-p_2-1} \left\{ \int_z^{+\infty} (x-z)^{p_2-1} \mu_G(x; f) dx \right\} dy \\
 &= M_2^{p_2} p_2 \int_0^{+\infty} y^{p-p_2-1} \left\{ \int_0^{+\infty} x^{p_2-1} \mu_G(x+z; f) dx \right\} dy \\
 &= M_2^{p_2} p_2 \int_0^{+\infty} y^{p-p_2-1} \left\{ \int_0^{+\infty} x^{p_2-1} \mu_G(x; f_2) dx \right\} dy \\
 &= M_2^{p_2} p_2 \int_0^{+\infty} \left\{ \int_0^{+\infty} x^{p_2-1} \mu_G(x; f_2) y^{p-p_2-1} dy \right\} dx \\
 &= M_2^{p_2} p_2 \int_0^{+\infty} \left\{ \int_{Ax^{\frac{1}{\xi}}}^{+\infty} x^{p_2-1} \mu_G(x; f_2) y^{p-p_2-1} dy \right\} dx \\
 &= M_2^{p_2} p_2 \int_0^{+\infty} x^{p_2-1} \mu_G(x; f_2) \left\{ \int_{Ax}^{+\infty} y^{p-p_2-1} dy \right\} dx \\
 &= \frac{A^{p-p_2}}{p_2-p} M_2^{p_2} p_2 \int_0^{+\infty} x^{p_2-1+p-p_2} \mu_G(x; f_2) dx \\
 &\leq \frac{A^{p-p_2}}{p_2-p} M_2^{p_2} p_2 \int_0^{+\infty} x^{p_2-1+p-p_2} \mu_G(x; f) dx.
 \end{aligned}$$

Collecting estimates (6.11), (6.12), (6.13) we see that integral in (6.8) does not exceed

$$(6.14) \quad M_1^{p_1} p_1 \frac{A^{p-p_1}}{p-p_1} \int_0^{+\infty} x^{p-1} \mu_G(x; f) dx \quad + \quad M_2^{p_2} p_2 \frac{A^{p-p_2}}{p_2-p} \int_0^{+\infty} x^{p-1} \mu_G(x; f_2) dx.$$

Now, using the identity

$$\int_0^{+\infty} x^{p-1} \mu_G(x; f) dx = \int_G |f(u)|^p du = \|f\|_{L^p(G)}^p,$$

and inequalities (6.8) and (6.14) we get

$$\|Af\|_{\ell^p(\widehat{G})}^p \leq \left(M_1^{p_1} p_1 \frac{A^{p-p_1}}{p-p_1} + M_2^{p_2} p_2 \frac{A^{p-p_2}}{p_2-p} \right)^p \|f\|_{\ell^p(\widehat{G})}^p.$$

Next we set

$$A = M_1^{\frac{p_1}{p_1-p_2}} M_2^{\frac{p_2}{p_2-p_1}}.$$

A simple computation shows that

$$M_1^{p_1} A^{p-p_1} = M_2^{p_2} A^{p-p_2} = M_1^{\frac{p_1(p_2-p)}{p_2-p_1}} M_2^{\frac{p_2(p_1-p)}{p_1-p_2}} = M_1^{1-\theta} M_2^\theta, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}.$$

Finally, we have

$$\|Af\|_{\ell^p(\widehat{G})} \leq K_{p,p_1,p_2} M_1^{1-\theta} M_2^\theta \|f\|_{L^p(G)},$$

where

$$K_{p,p_1,p_2} = \left(\frac{p_1}{p-p_1} + \frac{p_2}{p_2-p} \right)^{\frac{1}{p}}.$$

This completes the proof. □

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